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# Absence of Proper Nondegenerate Generalized Self-Similar Singularities

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We present a general covariance property and use it to prove that proper nondegenerate self-similar blow-up is not possible for active scalar equations.

**KEY WORDS:** Blow-up; Euler equations; active scalars; self-similar.

## 1. INTRODUCTION

We will discuss an active scalar equation[1, 2] and describe briefly the results of recent numerical simulations.[3] We will show that the active scalar exhibits a general path transformation covariance; then we use this property to prove that simple self-similar blow-up is not possible. The active scalar equation is a model for the incompressible Euler equations.

The three dimensional Euler equations are evolution equations for the three velocity components $u$,

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0 \tag{1}$$

coupled with a fourth equation, $\nabla \cdot u = 0$, expressing incompressibility. This is the Eulerian formulation, in which the velocity $u$ and pressure $p$ are recorded at fixed locations $x$. The velocities and pressure vanish at infinity. The pressure is determined using incompressibility.

The following are properties of the Euler equation that are shared *mutatis mutandis* by the active scalar: The equation is conservative, i.e., the

To Leo Kadakoff on the occasion of his 60th birthday.

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total kinetic energy, $\int |u|^2\,dx$ is a constant of motion. The possibility of singularity formation arises when one considers the time evolution of derivatives of $u$. The vorticity (anti-symmetric part of the gradient matrix) obeys a quadratic equation, the Helmholtz equation, that expresses the fact that vortex lines are material. The Helmholtz equation can be interpreted as the vanishing of a commutator

$$[D_t, \Omega] = 0 \tag{2}$$

where

$$D_t = \frac{\partial}{\partial t} + u \cdot \nabla$$

is the material derivative,

$$\Omega = \omega \cdot \nabla$$

and $\omega = \nabla \times u$. The characteristics of the first order differential operator $\Omega$ are the vortex lines, the characteristics of the material derivative $D_t$ are Lagrangian particle paths. The vorticity magnitude is the line element of the vortex line. It evolves according to the stretching equation

$$D_t(|\omega|) = \alpha |\omega| \tag{3}$$

The stretching factor $\alpha$ is related to the vorticity magnitude through a principal value singular integral:[1]

$$\alpha(x, t) = P.V. \int D(\hat{y}, \xi(x, t), \xi(x+y, t))\,|\omega(x+y, t)|\,\frac{dy}{|y|^d} \tag{4}$$

Here $d$ is the spatial dimension, $\hat{y}$ is the unit vector in the direction of $y$, $\xi(x, t) = \omega/|\omega|$ is the unit vector tangent to the vortex line passing through $x$ at time $t$ and $D$ is a specific geometric factor. The geometric factor is a smooth function of three unit vectors, has zero average on the unit sphere, $\int D\,dS(\hat{y}) = 0$ and vanishes pointwise when $\xi(x, t) = \pm\xi(x+y, t)$.

Because $\alpha$ has the same order of magnitude as $|\omega|$, dimensional reasoning predicts blow-up of the type one encounters in the ordinary differential equation $dm/dt = m^2$,

$$\sup_x |\omega(x, t)| \sim \frac{1}{T-t}$$

The well-known Beale–Kato–Majda criterion[4] guarantees that

$$\int_0^T \sup_x |\omega(x, t)|\, dt = \infty$$

is necessary for blow-up at time $T$.

## 2. ACTIVE SCALARS

We consider an active scalar $\theta = \theta(x, t)$ in $\mathbf{R}^2$ that solves

$$(\partial_t + u \cdot \nabla)\,\theta = 0 \tag{5}$$

where

$$u(x, t) = \int_{\mathbf{R}^2} \frac{1}{|x-y|} \nabla^{\perp}\theta(y, t)\, dy \tag{6}$$

This system resembles the Euler equation as was mentioned in the introduction. It is conservative: not only is $\int |u|^2\, dx$ a constant of motion but also $\int |\theta|^n\, dx$ are constants of motion for all $n$. The equation is equivalent to the requirement that a commutator vanishes as in (2): $D_t$ is the same material derivative and $\Omega = \omega \cdot \nabla$ with $\omega = \nabla^{\perp}\theta$. The characteristics of $\Omega$ are level lines of $\theta$. The Helmholtz equation for $\omega$, the stretching Eq. (3), the singular integral representation (4), the above mentioned properties of the geometric factor $D$ and the Beale–Kato–Majda criterion hold for the active scalar. In addition to this criterion, other more refined criteria exist for both the Euler equation[5] and the active scalar. These criteria are based on the representation (4) and, in particular, rule out the formation of a shock in $\theta$ across a smooth shock front.

A numerical study[2] proposed a certain initial datum and blow-up scenario. The presence of a hyperbolic saddle point in the graph of $\theta$ allowed for nonlinear growth of the gradient. One can imagine the local graph of $\theta$ as two hills that form an X-shaped range and change in time. The gradient growth is due to the steepening, as time passes, of the hill slopes. The maximum gradient is located close to the outside slopes and moves towards the vertex. As the slopes steepen, the aperture of the X shape closes. This depletes the temporal rate of change of the maximum gradient, consistent with the geometric depletion in $D$. A numerical study[6] suggested that the closing must happen in infinite time. In a recent rigorous and inspiring work D. Cordoba[7] proved that indeed this must be the case if the scalar level sets are locally distorted hyperbolas. The new numerical evidence[3] suggests that the maximum of the gradient does not blow up in finite time; the study supports the conclusion of proper non-degenerate self-similarity (see below).

## 3. GENERAL PATH TRANSFORMATIONS FOR ACTIVE SCALARS

Let $x = X(q, t)$ be a general path transformation—a time dependent, possibly nonvolume preserving diffeomorphism—of inverse $X^{-1}(x, t) = Q(x, t)$. Differentiating the identity

$$x = X(Q(x, t), t)$$

with respect to time $t$ and space $x$ and using the chain rule in (5) one can check that the function

$$\Theta(q, t) = \theta(X(q, t))$$

satisfies

$$(\partial_t + v \cdot \nabla)\, \Theta = 0 \tag{7}$$

where the *relabeling velocity* $v = v_X[u]$ of the path transformation $X$ is

$$v = v_X[u](q, t) = \left(\frac{\partial X(q, t)}{\partial q}\right)^{-1} \left(u(X(q, t), t) - \frac{\partial X(q, t)}{\partial t}\right) \tag{8}$$

We will call

$$U_X(q, t) = u(X(q, t), t)$$

the *path velocity* of $X$. One can show that

$$U_X(q, t) = \int_{\mathbf{R}^2} \frac{1}{|X(q, t) - X(p, t)|} \{\Theta(p, t), X(p, t)\}\, dp \tag{9}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial p_1}$$

In view of

$$\{f, g\}\, dp = df \wedge dg$$

and the fact that this expression is invariant under changes of variables it follows that the path velocity obeys

$$U_{X \circ Y} = U_X \circ Y \tag{10}$$

for general path transformations $Y$. Consequently the relabeling velocity obeys

$$v_{X \circ Y}[u](p, t) = \left(\frac{\partial Y(p, t)}{\partial p}\right)^{-1} \left(v_X[u](Y(p, t), t) - \frac{\partial Y(p, t)}{\partial t}\right)$$

that is

$$v_{X \circ Y}[u] = v_Y[v_X[u]] \tag{11}$$

Denoting $\theta_X = \Theta$ we have also, quite obviously

$$\theta_{X \circ Y} = (\theta_X)_Y \tag{12}$$

The above important general path transformation covariance properties are valid for all active scalar models and have counterparts in the 3D Euler equation. The usual Lagrangian path transformations are obtained by setting $v_X = 0$. The Eulerian path transformation is the identity, $X = I$, and, consequently, the Eulerian relabeling velocity equals the Eulerian path velocity. Thus, the Eulerian formulation of the active scalar problem uses a trivial path transformation and the Lagrangian formulation a trivial relabeling velocity.

The system (7), (8) and (9) can be viewed as a general path transformation formulation of the active scalar problem in the following manner. One is given a relabeling velocity

$$v(q, t) = v_X(q, t)$$

that might be a functional of $X$. Equation (7) is an active scalar equation that determines $\Theta$, assuming $X$ known. The path velocity $U_X$ is determined from $\Theta$ and $X$ through Eq. (9). The equation for the path transformation (8) can be written as an evolution equation that updates $X$

$$\frac{\partial X(q, t)}{\partial t} = U_X(q, t) - v(q, t) \cdot \nabla X(q, t) \tag{13}$$

This formulation is a generalization of both Eulerian and Lagrangian formulations. If $v(q, t)$ does not depend on $X$ then Eq. (7) is a linear, passive scalar equation and the formulation of the system is a version of the Lagrangian one.

## 4. PROPER NON-DEGENERATE SELF-SIMILARITY

Let us consider the case when $X(q,t)$ carries the singularities and $\Theta(q,t)$ is smooth. This is certainly the case for instance if $X$ is the Lagrangian path transformation and the initial datum $\theta(x,0)=\theta_0(x)$ is smooth: in that case $\Theta=\theta_0$. We are interested in the possibility of a non-Lagrangian transformation that captures the blow-up and has a relatively simple structure. For instance, one may assume a form

$$X(q,t)=Y(A(t)\,Z(q,t),t)$$

where both $Z(q,t)$ and $Y(p,t)$ are smooth diffeomorphisms up to and including the blow-up time and $A(t)$ is an invertible matrix that may blow up in finite time. one can see from the above general path transformation properties that we may consider, without loss of generality the case $Z=I$. Indeed, if $\Theta=\theta_{Y\circ A\circ Z}$ is smooth, then $\Theta_{Z^{-1}}=\theta_{Y\circ A}$ is also smooth.

We define thus generalized self-similarity the situation in which the true solution can be represented locally by

$$\theta(x,t)=\Theta(B(t)\,P(x,t),t) \tag{14}$$

where $\Theta$ is a smooth function of its arguments, $P(x,t)$ is likewise a smooth function from some fixed domain (open, connected) of the $x$ plane to a time varying domain of the plane, and the matrix $B(t)=(A(t))^{-1}$ is invertible but is allowed to blow up in finite time. This generalizes of course the familiar isotropic case where $P(x,t)=x$ and $B(t)$ is a scalar (multiple of the identity matrix). One should note that self-similar singularities, even if they exist formally, may have an exceptional character for the blow-up problem and not represent the typical dynamical formation of singularities. A case in point is the well-known one dimensional Burgers equation $\theta_t+\theta\theta_x=0$ where the self-similar ansatz $\theta=\Theta(x/(T-t))$ is consistent and leads to linear or constant profiles $\Theta$; nevertheless the formation of shocks from smooth data is not at all a self-similar phenomenon.

Using Eq. (13) and the ansatz $X=Y\circ A$ we obtain

$$\left(\frac{dA}{dt}\right)q=-Av(q,t)+v_Y(p,t) \tag{15}$$

where

$$v_Y(p, t) = v_Y(u)(p, t) = \left(\frac{\partial Y}{\partial p}\right)^{-1} \left(U_Y - \frac{\partial Y}{\partial t}\right)(p, t) \tag{16}$$

and $p = A(t)\, q$. Multiplying (15) from the left by $A(t)^{-1}$ and reading the result at $q = (A(t))^{-1}\, p$ we obtain the equation

$$\frac{dB(t)}{dt}\, p = v(q, t) - B(t)\, v_Y(p, t) \tag{17}$$

for

$$B(t) = A(t)^{-1}$$

We will say that the generalized self-similar ansatz is *non-degenerate* if there exists an invertible, symmetric matrix $\Lambda$, a positive function $\lambda(q, t) > 0$ and a positive constant $K$ such that

$$\nabla_q \Theta(q, t) = \lambda(q, t)\, \Lambda q \tag{18}$$

and

$$\left|\frac{\partial \Theta(q, t)}{\partial t}\right| \leqslant K\lambda(q, t) \tag{19}$$

hold. This definition corresponds to the situation in which $P(x, t) = 0$ describes the location of a nondegenerate critical point of $\theta$. (A critical point is a place where the gradient vanishes, a nondegenerate one is a critical point where the matrix of second derivatives is invertible.) An example of a nondegenerate self-similar ansatz is $\Theta(q_1, q_2, t) = F(q_1 q_2)$ with $F$ smooth and increasing; $B(t)$ and $P(x, t)$ can be arbitrary.

We will say that the nondegenerate self-similar blow-up ansatz (14) is *proper* if there exists positive constants $r$ and $C_0$ such that

$$\|B(t)\|^2 \leqslant C_0 \sup_{|p| \leqslant r} |B(t)\, p \cdot \Lambda B(t)\, p| \tag{20}$$

Here $\|B\|$ is a standard matrix norm. The inequalities (19) and (20) are required to hold throughout the time interval under consideration, up to the putative blow-up time. A continuous function is said to be proper if the preimages of all compacts are compacts. The function $\Theta = q_1^2 + q_2^2$ is proper, the function $\Theta = q_1 q_2$ is not. Not all proper functions give rise to proper generalized self-similar blow-up scenarios. But also, a nonproper function can be the profile of a proper generalized self-similar blow-up ansatz. A relevant example corresponds to a compression of one direction and the collapse of one arm of a locally hyperbolic saddle onto the other arm. Specifically, $\Theta(q, t) = F(q_1 q_2, t)$ with $F$ smooth and increasing and $B(t)$ is given by

$$q_1 = p_1 - M(t)\, p_2$$
$$q_2 = \Gamma(t)\, p_2$$

The non-negative coefficient $M(t)$ describes the collapse; at $t = 0$ it equals zero and at subsequent times it grows. The line $q_1 = 0$ starts out as the line $p_1 = 0$, at later times corresponds to the line $p_2 = [1/M(t)]\, p_1$ and collapses on the line $p_2 = 0$ when $M \to \infty$. We assume that $\Gamma(t) \geqslant 1$ to indicate compression in the $p_2$ direction. $\Theta$ has a nondegenerate hyperbolic saddle at the origin and is not a proper function, but the blow-up ansatz is a proper blow-up. Indeed, in order to verify that (20) holds we have to bound the individual matrix elements in terms of an upper bound for $|q_1 q_2|$ on a whole neighborhood of the origin in the $p$ plane. By choosing a point with $p_1 = 0$ we obtain a bound for $M(t)\, \Gamma(t)$. Then, by choosing a point with $p_1 = p_2 \neq 0$ we obtain an upper bound for $\Gamma(t)$.

We will prove that a proper, nondegenerate generalized self-similar blowup cannot happen in finite time for the active scalar. We take the scalar product of (17) with $AB(t)\, p$. Making use of (18) and of (7) we obtain

$$\left(\frac{dB(t)}{dt} p \cdot AB(t)\, p\right) = -\frac{1}{\lambda(q, t)} \frac{\partial \Theta}{\partial t}(q, t) - B(t)\, v_Y(p, t) \cdot AB(t)\, p \tag{21}$$

where $q = B(t)\, p$.

We will use a bound on the supremum of $|u|$. The relationship between $\theta$ and $u$ is such that the sup norm of $u$ fails only logarithmically to be bounded by the sup norm of $\theta$. More precisely, let us define the length $L$ by

$$L^2 = \frac{\|\theta\|_{L^1(\mathbf{R}^2)}}{\|\theta\|_{L^\infty(\mathbf{R}^2)}} \tag{22}$$

Because of the conservation of all $L^p$ norms the quantity $L$ does not change in time along solutions of the system (5). A known inequality[8] states that there exists an absolute constant $C$ so that the integral (6) satisfies

$$\|u\|_{L^\infty(\mathbf{R}^2)} \leqslant C\left(1+\log_+\left(\frac{L\,\|\nabla\theta\|_{L^\infty(\mathbf{R}^2)}}{\|\theta\|_{L^\infty(\mathbf{R}^2)}}\right)\right)\|\theta\|_{L^\infty(\mathbf{R}^2)} \tag{23}$$

for all functions $\theta$. ($\log_+ x = \max(\log x, 0)$.) A direct consequence of this inequality is

$$\sup_p |v_Y(p,t)| \leqslant C_1(1+\log_+\|B(t)\|) \tag{24}$$

We take $p$ satisfying $|p| \leqslant r$ with $r$ given in the definition of a *proper* generalized self-similar ansatz. We integrate (21) in time, using the inequalities (19) and (24). we then take the supremum for all such $p$ and use the inequality in (20). We obtain

$$\|B(t)\|^2 \leqslant C_2 \int_0^t (1+\log_+(\|B(s)\|))\,\|B(s)\|^2\,ds + C_3$$

One can deduce from this inequality that $\|B(t)\|$ grows at most as a double exponential, and in view of the proper nondegenerate self-similar ansatz so does the maximum of gradient of the scalar.

## 5. CONCLUSION

We have ruled out the possibility of a finite time proper nondegenerate generalized self-similar blow-up in a simple incompressible active scalar. The proof follows from a general covariance of the equation; the relative simplicity of the ansatz is used also. Blow-up of this kind includes as a particular case certain nonisotropic singularity formation scenarios that were suggested by direct numerical simulations.

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